0.1 Context

variety: $X \subset \mathbf{P}^n$
sheaf: Ω^r_X
Hodge numbers: $h^{p,q} = \dim(H^q(X, \Omega^p_X))$
Hodge diamond:

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ \iddots & & \vdots & & \ddots \\ & & h^{n,n} & & \end{array}$$

L-R symmetry: $H^{p,q} = \overline{H^{q,p}}$
T-B symmetry: Hard Lefschetz Theorem or Hodge star operator

X is a hypersurface defined by a degree d polynomial $\implies$ X is a very ample effective divisor on $\mathbf{P}^n$

Here X is a variety in $\mathbf{P}^n$. Recall Ω^p_X is the sheaf of differential p-forms on X, or the cotangent sheaf. We write $h^{p,q} = \dim(H^q(\Omega^p_X))$ for the Hodge numbers, and when X is a Kähler manifold, we have the Hodge decomposition

$$H^k(\mathbf{P}^n) = \bigoplus_{p+q=k} H^{p,q}.$$

All smooth projective varieties are Kähler, as $\mathbf{P}^n$ is Kähler (restrict metric). A Hodge diamond starts with

$$\begin{array}{ccc} & h^{0,0} & \\ h^{1,0} & & h^{0,1} \\ \iddots & \vdots & \ddots \end{array}$$

and continues on until $h^{n,n}$, where n is the complex dimension of the variety. Left-right symmetry because $H^{p,q} = \overline{H^{q,p}}$ (complex conjugation). Top-bottom symmetry from either Hard Lefschetz theorem, map induced by Hodge star operator, or Poincare duality (vanishing of cup product on forms).

In this talk X is a hypersurface in $\mathbf{P}^N$ defined by a degree d polynomial. We may also consider X as a very ample effective divisor on X, which allows us to apply the Lefschetz hyperplane theorem, necessary for finding almost all the Hodge numbers of X.

0.2 Hodge diamond of $\mathbf{P}^n$

$$h^k(\mathbf{P}^n) = \begin{cases} 1 & k \text{ even}, \leqslant 2n, \\ 0 & k \text{ odd} \end{cases}$$

DRAW HODGE DIAMOND

LHT (1924): Let X be a smooth projective variety of dimension n, and D an ample effective divisor on X. Then $H^k(X) \to H^k(D)$ is an iso for $k < n-1$ and an injection for $k = n-1$.

Recall the regular cohomology of $\mathbf{P}^n$ is $H^k(\mathbf{P}^n) = \mathbf{Z}$ if k is even and 0 if k is odd. So every other has to sum up to 1 or to 0. By left-right symmetry, ones down the middle, zeros everywhere else.

Recall the Lefschetz Hyperplane theorem from Seckin's talk. Since a hypersurface in $\mathbf{P}^n$ is a very ample effective divisor on $\mathbf{P}^n$, we get *everything* except middle row in Hodge diamond for any hypersurface. Note X has dimension $n-1$.

For small dimensions, use tricks to get numbers. For large dimensions, chase SESs or use Hirzebruch–Riemann–Roch.

0.3 Example: the quintic threefold

$n = 4$
$d = 5$
X is a quintic threefold in $\mathbf{P}^4$

Let $n = 4$ and consider a hypersurface cut out by a degree 5 polynomial in $\mathbf{P}^4$. We know almost all the Hodge diamond, except for the middle row.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ \alpha & & \beta & & \beta & & \alpha \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

First let's find $\alpha = h^{3,0} = \dim(H^0(\Omega^3_X)) = \dim(H^0(\omega_X))$, for ω_X the canonical bundle of X. We need to find $\omega_{\mathbf{P}^n}$ because it will be useful later (and is easier than ω_X).

This is easy to find - choose a basis, calculate transition functions between patches, find they have pole of order $n+1$ at ∞. Hence $\omega_{\mathbf{P}^n} \cong \mathcal{O}_{\mathbf{P}^n}(-n-1)$.

$\alpha = h^{3,0} = \dim(H^0(\Omega^3_X)) = \dim(H^0(\omega_X))$

We find $\omega_{\mathbf{P}^n} = \mathcal{O}_{\mathbf{P}^n}(-n-1)$
$\omega_X = \mathcal{O}_X(-n-1+d)$

By doing similar computations (and the chain rule), we get an extra factor z_1^d in the transition functions, meaning they have a pole of order $n+1-d$ at ∞. Hence $\omega_X \cong \mathcal{O}_X(-n-1+d)$.

Take a free resolution of $\mathcal{O}_X$:

This resolution of $\mathcal{O}_X$ is just the cokernel of multiplying by F, the degree d polynomial defining X.

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n}(-d) \xrightarrow{\cdot F} \mathcal{O}_{\mathbf{P}^n} \xrightarrow{\mathrm{cok}(\cdot F)} \mathcal{O}_X \longrightarrow 0$$

Twist by $-n-1+d$:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\mathbf{P}^n}(-n-1) \arrow[r] \arrow[d, phantom, "\wr\Vert"] & \mathcal{O}_{\mathbf{P}^n}(-n-1+d) \arrow[r] & \mathcal{O}_X(-n-1+d) \arrow[r] \arrow[d, phantom, "\wr\Vert"] & 0 \\ & \omega_{\mathbf{P}^n} & & \omega_X & \end{tikzcd}$$

$H^0(\mathcal{O}_{\mathbf{P}^n}(-n-1+d)) \cong H^0(\omega_X)$

$$h^0(\mathcal{O}_{\mathbf{P}^n}(k)) = \left(\begin{array}{c} \text{\# of homogeneous polynomials} \\ \text{of degree } k \text{ in } n \text{ variables} \end{array} \right) = \binom{k+n}{n}$$

$\implies h^0(\omega_X) = \binom{d-1}{n}$
$\implies \alpha = 1$

Now we have a free resolution of ω_X. Since resolution, everything zero except last two. Get isomorphism between H^0 of the last two objects. We are only looking for the dimension of this space.

We know the dimension of $H^0(\mathcal{O}_{\mathbf{P}^n}(k))$ is the number of homogeneous polynomials of degree k in n variables SEE ALG GEO NOTES, which is $\binom{k+n}{n}$. In this case we have $\binom{d-1}{n} = \binom{4}{4} = 1$, hence $\alpha = 1$.

To find β:

Theorem 0.3.1. For X a hypersurface in $\mathbf{P}^n$,

$$\deg(c_{n-1}(T_X)) = \chi_{top}(X) = \sum_{i=0}^{2n} \dim(H^i(X)).$$

Hence $4 - 2\alpha - 2\beta = \chi_{top}(X)$.

So if we know $\chi_{top}(X)$, then we will also know β. Use two sequences:

$$\text{Euler sequence of } \mathbf{P}^n\text{:} \quad 0 \longrightarrow \mathcal{O}_{\mathbf{P}^n} \longrightarrow \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus n+1} \longrightarrow T_{\mathbf{P}^n} \longrightarrow 0$$

$$\begin{array}{l} \text{Normal bundle sequence of } X\text{:} \quad 0 \longrightarrow T_X \longrightarrow T_{\mathbf{P}^n}|_X \longrightarrow N_X \longrightarrow 0 \\ \hphantom{\text{Normal bundle sequence of } X\text{:} \quad 0 \longrightarrow T_X \longrightarrow T_{\mathbf{P}^n}|_X \longrightarrow\ } \| \\ \hphantom{\text{Normal bundle sequence of } X\text{:} \quad 0 \longrightarrow T_X \longrightarrow T_{\mathbf{P}^n}|_X \longrightarrow} \mathcal{O}_X(d) \end{array}$$

$H^*(\mathbf{P}^n) = \mathbf{Z}[t]/t^{n+1}$
By Whitney sum formula:

$$\begin{aligned} c(T_{\mathbf{P}_n})c(\mathcal{O}_{\mathbf{P}^n}) &= c(\mathcal{O}_{\mathbf{P}^n}(1)^{\oplus n+1}) \\ c(T_{\mathbf{P}^n}) &= c(\mathcal{O}_{\mathbf{P}^n}(1))^{n+1} \\ &= (1+t)^{n+1}, \end{aligned}$$

Hence $c(T_{\mathbf{P}^n}|_X) = (1+u)^{n+1}$

$$\begin{aligned} c(T_X) = \frac{(1+u)^{n+1}}{c(\mathcal{O}_X(d))} &= \frac{(1+u)^{n+1}}{1+du} \\ &= \sum_{i=0}^{n+1} \binom{n+1}{i} u^i \sum_{j=0}^{\infty} (-du)^j, \end{aligned}$$

$$\begin{aligned} \chi_{top}(X) &= \deg(c_3(T_X)) \\ &= \langle c_3(T_X), [X] \rangle \\ &= \sum_{i=0}^{3} \binom{5}{i} (-d)^{3-i} u^3[X] \\ &= \sum_{i=0}^{3} \binom{5}{i} (-1)^{3-i} d^{4-i} \\ &= -d^4 + 5d^3 - 10d^2 + 10d \\ &= -250 + 50 \\ &= -200 \end{aligned}$$

$\implies 4 - 2\alpha - 2\beta = -200$
$\implies \beta = 101.$

The total Chern class of middle term of short exact sequence is product of end terms. Let t be the generator of cohomology ring of $\mathbf{P}^n$. Via projection map $p : X \to \mathbf{P}^n$, we get $u = p_*^{-1}(t)$ as a generator of the cohomology ring of X.

We also used the fact that $N_X \cong \mathcal{O}_X(d)$ since X is a hypersurface cut out by a degree d polynomial in $\mathbf{P}^n$. Note that $c_k(T_X)$ is the term in $c(T_X)$ with u^k in it.

The degree of c_k is the evaluation of c_k on the fundamental class $[X] \in H^3(X)$, the top cohomology class. We used the fact that $u^3[X] = d$, since $H_3(X)$ is 1-dimensional.

0.4 General approach

In general, define:

$$h^{p,q} = \dim(H^q(X, \Omega_X^p))$$
$$\chi^p = \sum_{q=0}^{n-1} (-1)^q h^{p,q}$$
$$\chi_y = \sum_{p=0}^{n-1} \chi^p y^p$$

Then:

$$\chi_y = [z^n] \frac{1}{(1+zy)(1-z)^2} \cdot \frac{(1+zy)^d - (1-z)^d}{(1+zy)^d + y(1-z)^d}$$

Example:
Quintic threefold ($n = 4, d = 5$): $\chi_y = 100y - 100y^2$
Cubic surface ($n = 3, d = 3$): $\chi_y = 1 - 7y + y^2$

For any hypersurface in $\mathbf{P}^n$, we know all but the middle row by LHT, so $h^{p,q} = \delta_{p,q}$ whenever $p + q \neq n$. The Hirzebruch–Riemann–Roch theorem allows us to calculate exactly this middle row, by giving an alternating sum of the entries in the diagonals of the Hodge diamond.

This method also works for all complete intersections - non-singular surfaces in general position. Also works for sheaf cohomology of sheaf of differential forms tensored with line bundles.

CAS can find this coefficient efficiently. Look in Hirzebruch or talk to Ben.