1 Setting

Def: The *n-sphere* is $S^n = \{x \in \mathbf{R}^{n+1} \ : \ |x| = 1\}$, for $n \geqslant 0$.

Def: The kth homotopy group of S^n is $\pi_k(S^n) = \{\text{continuous maps } f : S^k \to S^n\}/\text{homotopy}$, for $k \geqslant 0$.

Example: $\begin{matrix} \pi_1(S^1) = \mathbf{Z} & \pi_1(S^2) = 0 \\ \pi_2(S^1) = 0 & \pi_2(S^2) = \mathbf{Z} \end{matrix}$

Unknown (i.e. $\pi_k(S^n)$) in general. But there are some patterns.

2 Stability

Def: The *suspension* of a topological space X is $\Sigma X = X \times I / X \times \{0\}, X \times \{1\}$. There is an induced homomorphism $\sigma : \pi_r(X) \to \pi_{r+1}(\Sigma X)$.

Example: $\Sigma S^n = S^{n+1}$ for $n \geqslant 0$. The induced homomorphism is $\pi_r(S^n) \to \pi_{r+1}(S^{n+1})$.

Thm: (Freudenthal suspension theorem, 1937)
For $k \geqslant 0$, the induced homomorphism $\sigma : \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ is an isomorphism for $n > k+1$ and a surjection for $n = k+1$.

Def: The *kth stable homotopy group* of the sphere is $\pi_k^S = \pi_{n+k}(S^n)$ for $n > k+1$.

SHOW LARGE HOM GRP TABLE

Below the bold black line the Freudenthal suspension theorem holds (look at diagonals). Observation:

Thm: (Serre, 1951)
$\pi_k(S^n)$ is finite abelian except for $\pi_n(S^n)$ and $\pi_{4n-1}(S^{2n})$, when it is $\mathbf{Z} \oplus F$ for F finite abelian.

But how to compute higher $\pi_k(S^n)$? Use the *Serre spectral sequence.*

Thm/Def: For any group A, the *Eilenberg–MacLane space* $K(A, n)$ has

$$\pi_k(K(A,n)) = \begin{cases} A & \text{if } k = n, \\ 0 & \text{else.} \end{cases}$$

If a topological space X has $\pi_n(X) = A$, then there exists a map $f : X \to K(A, n)$ such that

$$\pi_n(f) : \pi_n(X) \to \pi_n(K(A,n)) \qquad , \qquad H_n(f) : H_n(X) \to H_n(K(A,n))$$

are isomorphisms.

Set $F = f^{-1}(p)$ for any $p \in K(A, n)$ to get fiber sequence $F \to X \to K(A, n)$. Then

$$\pi_i(F) = \begin{cases} \pi_i(X) & \text{if } i > n, \\ 0 & \text{else.} \end{cases}$$

Since we know $H_*(K(A,n))$, use SSS to compute $H_*(F)$.

Serre spectral sequence

Adams spectral sequence

Adams–Novikov spectral sequence