A brief survey of K3 surfaces

Jānis Lazovskis

Abstract: This monograph gives a view, from both differential and algebraic geometry, of $K3$ surfaces. First we build-up to the Riemann–Roch theorem on $K3$ surfaces, the statement of which needs several powerful tools, including divisors and sheaves, which are introduced along the way. Next we describe in detail the cohomology groups and Hodge decomposition of $K3$ surfaces, finishing with some current research directions.

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Introduction

A $K3$ *surface* is a 2-dimensional complex manifold M with trivial canonical bundle and $H^1(M, \Omega^2(M)) = 0$. They were first studied by André Weil in 1957, and he very quickly became convinced that $K3$ surfaces are all part of a single "family" (proven in 1964 by Kunihiko Kodaira) and that they admit Kähler metrics (proven in 1983 by Yum-Tong Siu). These are two of the most interesting properties of $K3$ surfaces, given here as Assertions 1.1.4 and 1.1.5. For more history on the formative years of the topic, see [Buc03].

The modern history of $K3$ surfaces closely follows that of Calabi–Yau manifolds (proven to exist in 1976 by Shing-Tung Yau), as $K3$ surfaces are one of two examples of such manifolds in dimension 2 (the other being complex tori). Physicists use Calabi–Yau manifolds for their models, since the properties match up exactly with what the physicists need.

There exist algebraic (very easy to describe) and non-algebraic (very difficult to describe) $K3$ surfaces, of which we will study the former. A powerful consequence of the fact that all $K3$ surfaces are diffeomorphic is that we need only study the ones that are easy to describe to understand all $k3$ surfaces.

1 Geometric setting

1.1 Understanding $K3$ surfaces

Definition 1.1.1.

· a *complex n-manifold* M is a manifold with charts $\varphi : U \to \mathbf{C}^n$

· a *holomorphic bundle* on M is a vector bundle with E complex and $\pi : E \to M$ holomorphic

canonical bundle · the *canonical bundle* of M is $\omega_M := \Omega^n(M)$, a line bundle generated by $dx_1 \wedge \cdots \wedge dx_n$

Recall $\Omega^p(M) = \bigwedge^p T*M = (T^*M)^{\otimes p}/I$ for I the ideal of elements $v^{\otimes k}$, $k \geqslant 2$. This gives the vector bundle structure for $\Omega^n(M)$.

variety When it is convenient, we view manifolds in the differential geometric sense equivalently as smooth *algebraic varieties*. A smooth algebraic variety is the set of points in a n-dimensional vector space that satisfy equations $f_1 = 0, \ldots, f_k = 0$, where all the $f_i : \mathbf{R}^n \to \mathbf{R}$ are smooth. That is, none of the f_i have *singularities*, or points where the derivative is not defined.

projective space Further, when it is convenient (and possible), we may view all our manifolds and varieties as sitting in *projective space*, which means a point (x, y, z, w) is the same as $(\lambda x, \lambda y, \lambda z, \lambda w)$, for all non-zero λ. Since complex projective space, or $\mathbf{CP}^n$ is an n-dimensional manifold, we may view manifolds in projective space as submanifolds of $\mathbf{CP}^n$. It is convenient (and customary) to denote an element in $\mathbf{CP}^n$ by $[x_0 : x_1 : \cdots : x_n]$, and to write simpy $\mathbf{P}^n$.

Recall a *homogeneous* polynomial of degree d in n variables is of the form

$$\sum_{i_1+\cdots+i_n=d} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}.$$

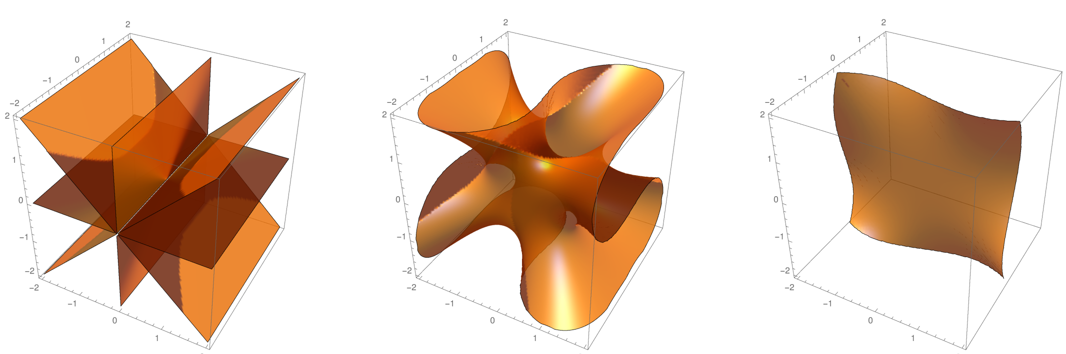
Proposition 1.1.2. The zero locus of any smooth homogeneous degree 4 equation is a $K3$ surface.

Proof: (Sketch) Let $C = \{(x, y, z, w) \; : \; f(x, y, z, w) = 0\}$ for f homogeneous in degree 4. Hence we may embed it in $\mathbf{P}^3$ (as it has 4 variables), which is 3-dimensional. One equation increases codimension by 1, and a codimension 1 space in $\mathbf{P}^3$ is 2-dimensional. Next, C is a manifold because the graph of a function on n variables may be projected injectively onto $\mathbf{C}^n$, giving the atlas.

For the canonical bundle, a theorem from algebraic geometry states that $\omega_C = \mathcal{O}_C(-n-1+d)$ for a manifold in $\mathbf{P}^n$ defined by a degree d homogeneous equation. So in this case, we have $\mathcal{O}(-3-1+4) = \mathcal{O}_C(0)$, which is the trivial bundle (see Definition 1.3.2).

For the cohomology, a smooth hypersurface in $\mathbf{P}^3$ is simply connected, which allows us to apply the Lefschetz hyperplane theorem. The theorem gives that $\pi_i(C) \cong \pi_i(\mathbf{P}^n)$ for $i < n/2$, so especially for $i = 1$. Since $\pi_1(\mathbf{P}^3) = 0$, and $H_1(C, \mathbf{C})$ is the abelianization of $\pi_1(\mathbf{C})$, we get $H_1(C, \mathbf{C}) = 0$. Since $H^1(C, \Omega^2(C))$ is half this number (by Theorems 2.1.1 and 1.4.3), which is zero, $H^1(C, \Omega^2(C)) = 0$. ■

Example 1.1.3. The zero locus of $f(x, y, z, w) = x^4 + y^4 + z^4 + w^4$ is a $K3$ surface. This is called the *Fermat quartic*. Below are some projections to $\mathbf{R}^3$ of this surface.



To make some of the proofs further on easier, we assume two very strong statements.

Assertion 1.1.4. Any two $K3$ surfaces are diffeomorphic.

In fact, any two $K3$ surfaces are *deformation equivalent*, an even stronger condition. See Theorem 8.6 in Chapter VIII of [Bar+04] for more on this. We will not even get anywhere near proving this theorem, but it will be useful for the Hodge decomposition theorem.

Assertion 1.1.5. $K3$ surfaces are Kähler manifolds.

Kähler manifold

A *Kähler manifold* is a complex manifold with a certain inner product on tangent spaces that varies smoothly, with an associated closed 2-form. Kähler geometry is a very rich field, but we will not discuss it here.

Next we introduce some terms from algebraic geometry.

1.2 Divisors and line bundles

divisor

Definition 1.2.1. Let X be a variety. A *divisor* on X is a formal finite linear combination

$$D = \sum_{i=1}^{r} k_i C_i$$

of codimension-1 subvarieties C_i of X and $k_i \in \mathbf{Z}$.

Recall the *codimension* of a variety (rather, one way to define the codimension) is the number of independent equations used to define it, and a *subvariety* is a variety contained within another variety.

Example 1.2.2. Let X be the zero locus of $x^4 + y^4 + z^4 + w^4$, as above. Define

$$\begin{aligned} C_1 &= \{(x, y, z) \in X \ : \ x^2 = 0\}, \\ C_2 &= \{(x, y, z) \in X \ : \ x + y + z = 0\}, \end{aligned}$$

which are codimension 1 subvarieties in X, as each is defined by a single equation. Then $D = 5C_1 - 3C_2$ is a divisor on X.

Definition 1.2.3. Let X be an n-dimensional variety, C a codimension 1 subvariety of X, and f a meromorphic function on X (that is, $f \in \mathbf{C}(X)$). Write

$$v_C(f) := \begin{cases} k > 0 & \text{if } f \text{ has a zero of order } k \text{ along } C, \\ -k < 0 & \text{if } f \text{ has a pole of order } k \text{ along } C. \end{cases}$$

Then $\text{div}(f) = \sum_{C \subset X} v_C(f)$ is the *divisor associated to* f.

Definition 1.2.4. Let X be a variety and $D = \sum_i k_i C_i$, $D' = \sum_i \ell_i C_i$ two divisors on X. Then
- D is *principal* if $D = \text{div}(f)$ for $f \in \mathbf{C}(X)$;
- D is *effective* if $k_i \geqslant 0$ for all i, and we write $D \geqslant 0$;
- D is *linearly equivalent* to D' if $D - D' = \text{div}(f)$ for some f.

In the context of 2-dimensional complex manifolds in $\mathbf{P}^3$, there are no non-constant globally-defined meromorphic functions, so there a principal divisor is always 0.

section

Recall that a vector bundle $\pi : E \to X$ on a complex manifold X had $\pi^{-1}(U) \cong U \times \mathbf{C}^n$, and a *section* of the bundle was a map $s : U \to E$ such that $\pi \circ s = \text{id}_U$. Every section has a zero locus, to which a divisor may be associated. That is, if the section has a zero of multiplicity k along a subvariety C, then a term kC appears in the associated divisor.

For line bundles, when $n = 1$, a section is just a choice of $c \in \mathbf{C}$, since $s(x) = (x, c) \in U \times \mathbf{C}$. We can get from a non-zero c_1 to c_2 by multiplying by c_2/c_1, so the difference between any two sections is just scalar multiplication. Hence any two divisors on a line bundle are linearly equivalent.

Picard group With this correspondence, the tensor product operation on divisors is clear (it is just the tensor product on the associated line bundles). The *Picard group* of a variety X, denoted $\mathrm{Pic}(X)$, is the group of linearly equivalent divisors with group operation tensor product. This group will become useful when we want to compare varieties, as it is a (relatively strong) invariant of X.

1.3 Sheaves

sheaf **Definition 1.3.1.** Let X be a variety. Let X be a topological space. A *sheaf* $\mathcal{F}$ on X is an assignment of an abelian group $\mathcal{F}(U)$ to every open $U \subset X$ and maps $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for all $V \subset U$, such that

1. $\rho_{UU} = \mathrm{id}$,
2. $\rho_{VW}\rho_{UV} = \rho_{UW}$ whenever $W \subset V \subset U$,
3. for any families $\{U_\alpha\}_{\alpha\in A}$ and $\{s_\alpha\}_{\alpha\in A}$, where $s_\alpha \in \mathcal{F}(U_\alpha)$, if

$$\rho_{U_\alpha, U_\alpha\cap U_\beta}(s_\alpha) = \rho_{U_\beta, U_\alpha\cap U_\beta}(s_\beta)$$

for all $\alpha, \beta \in A$, then there exists a unique $s \in \mathcal{F}(U = \bigcup_{\alpha\in A} U_\alpha)$ such that $\rho_{UU_\alpha}(s) = s_\alpha$.

The last condition is known as the *gluing axiom*. Objects that satisfy the first two but not the gluing axiom are called *presheaves*, though it is always possible to associate a sheaf to a presheaf, through a process called *sheafification*.

$\mathcal{O}_X$ **Definition 1.3.2.** The *sheaf of homolorphic functions* on a complex variety X is denoted $\mathcal{O}_X$. The restriction map ρ_{UV} is the regular restriction map $f|_V$, and the gluing axiom is satisfied, as being holomorphic is a local property.

For X a variety with k connected components, $\ker(d(\mathcal{O}_X)) \cong \mathbf{C}^k$.

invertible sheaf Given sheaves $\mathcal{F}, \mathcal{G}$ over the same space X, it is possible to create new sheaves $\mathcal{F} \oplus \mathcal{G}$, $\mathcal{F} \otimes \mathcal{G}$, $\mathcal{F}^*$, etc, with these and other algebraic operations. We say a sheaf $\mathcal{F}$ is *invertible* if there exists a sheaf $\mathcal{G}$ such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$. Recall that the canonical bundle on a $K3$ surface is trivial, so $\omega_X \cong \mathcal{O}_X$ (see below for bundle and sheaf correspondences).

A vector bundle is a special type of sheaf, where instead of $\pi^{-1}(U) \cong U \times V$ locally, for V some vector space, we have $\pi^{-1}(U) \cong U \times A$, for A some abelian group. A *locally free* sheaf is the same as a vector bundle. Hence we use the words "canonical bundle" and "*canonical sheaf*" interchangeably.

Definition 1.3.3. Let X be a variety and D a divisor on X. Then

$$\mathcal{O}_X(D) = \{f \in \mathbf{C}(X) \ : \ \mathrm{div}(f) + D \geqslant 0\}$$

is a subsheaf of $\mathcal{O}_X$, called the *invertible sheaf associated to* D. Sometimes it is denoted $\mathcal{L}(D)$.

Before we get into cohomology, we need a few more adjectives for sheaves.

Definition 1.3.4. Let $\mathcal{F}$ a sheaf on a variety X. Then
· an *injective resolution* of $\mathcal{F}$, denoted $\mathcal{F}^\bullet$, is a particular sequence of sheaves $0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \cdots$;
· a *global section* of $\mathcal{F}$ is a nowhere vanishing element of $\mathcal{F}(X)$.

The functor Γ that takes in sheaves and spits out global sections is called the *global section functor*.

1.4 The Riemann–Roch theorem

The goal of this section is to understand the statement of the Riemnn–Roch theorem (a proof will not be given, since it is quite long and involved) so that we may use it in the next section.

de Rham cohomology

Definition 1.4.1. Let M be a complex n-manifold and $\Omega^r(M) := \bigwedge^r T^*M$ the space of differential r-forms on M. Then the *de Rham cohomology* of M is the cohomology of the sequence

$$0 \longrightarrow \Omega^1(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(X) \longrightarrow 0,$$

with coefficients in $\mathbf{C}$, where d is the formal derivative.

Definition 1.4.2. Let X be a variety and $\mathcal{F}$ a sheaf on X. The *ith cohomology group of* $\mathcal{F}$ is the ith cohomology group of $\Gamma(X, \mathcal{F}^\bullet)$. That is, $H^i(X, \mathcal{F}) := H^i(\Gamma(X, \mathcal{F}^\bullet))$.

The above definition needs some unpacking to get at what is going on behind the scenes. In practice, we rarely use injective resolutions to find the cohomology with coefficients in a sheaf, since such resolutions are difficult to find. Instead we use slick duality theorems.

Serre duality

Theorem 1.4.3. [Serre]
Let $\mathcal{F}$ be a locally free sheaf over an n-dimensional algebraic variety X. Then for all $0 \leqslant k \leqslant n$,

$$H^k(X, \mathcal{F}) = H^{n-k}(X, \mathcal{F}^* \otimes \omega_X)^*.$$

dual sheaf

The above theorem is known as *Serre duality*, and it uses the idea of a *dual sheaf*, defined as $\mathcal{F}^* := \mathrm{Hom}(\mathcal{F}, \mathcal{O}_X)$. This theorem will be applied almost exclusively when $\mathcal{F} = \mathcal{O}_X$, so $\mathcal{O}_X^* = \mathcal{O}_X$. From this we get more information about $K3$ surfaces.

Proposition 1.4.4. Let X be a $K3$ surface. Then $\dim(H^1(X, \mathcal{O}_X)) = 0$ and $\dim(H^2(X, \mathcal{O}_X)) = 1$.

Proof: The first statement is from the definition of a $K3$ surface. For the second statement, apply Serre duality with $k = 2$ and $n = 2$. Since $\mathcal{O}_X^* \otimes \omega_X \cong \mathcal{O}_X^* \cong \mathcal{O}_X$, as ω_X is the identity element in the Picard group, and $\mathcal{O}_X$ is its own inverse. Hence $H^2 = H^0 = \mathbf{C}$, whose dimension is 1. ■

The next theorem is not required for the statement or proof of the Riemann–Roch theorem, but it is quite useful and will be applied for the main result of the next section. The (co)homologies used are the regular singular/cellular versions.

Poincaré duality

Theorem 1.4.5. [Poincaré]
Let M be a compact oriented complex n-manifold. Then $H^k(M, \mathbf{C}) \cong H_{n-k}(M, \mathbf{C})$ for all $0 \leqslant k \leqslant n$.

Definition 1.4.6. Let $\mathcal{F}$ be a sheaf over a variety X. The *holomorphic Euler characteristic* of $\mathcal{F}$ is

$$\chi(\mathcal{F}) := \sum_{i=0}^{\dim(X)} (-1)^i \dim(H^i(X, \mathcal{F})).$$

Example 1.4.7. Let X be a topological space and $\mathcal{F}$ the trivial bundle over X. Then $\chi(\mathcal{F}) = \chi(X)$, the topological Euler characteristic of X. For instance, if X is a 2-dimensional space that has been triangulated, then $\chi(X) = (\text{number of vertices}) - (\text{number of edges}) + (\text{number of faces})$.

intersection number

Definition 1.4.8. Let X be a variety and L_1, L_2 two line bundles over X. The *intersection number* of L_1 and L_2 is, equivalently, any of the following expressions:

$$\begin{aligned} L_1.L_2 &:= \chi(\mathcal{O}_X) - \chi(L_1^*) - \chi(L_2^*) + \chi(L_1^* \otimes L_2^*) && \text{(sum of Euler characteristics)} \\ &= [nm]\chi(X, L_1^n \otimes L_2^m) && \text{(coefficient in a polynomial)} \\ &= \left(\begin{array}{c} \text{number of times, with multiplicity, the} \\ \text{associated divisors of } L_1 \text{ and } L_2 \text{ intersect} \end{array} \right). && \text{(natural interpretation)} \end{aligned}$$

For more on why these are equivalent, see [Huy15], Chapter 1.1.

The way we've defined the intersection number above, it is not clear what an "intersection" of divisors is. Since it is a difficult task to define this precisely and needs many more tools from algebraic geometry, we use the heurstic interpretation of curves intersecting on a surface. In general, the intersection number may be viewed as a symmetric bilinear map $\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbf{Z}$, lending to the usefulness of the Picard group.

We are now ready to understand the statement of the Riemann–Roch theorem.

Theorem 1.4.9. [Riemann, Roch]
Let X be a $K3$ surface and L a line bundle over X. Then

$$\chi(L) = \frac{1}{2}L.L + 2.$$

For an arbitrary complex projective surface X, the theorem says $\chi(L) = \frac{1}{2}L.(L - \omega_X) + \chi(\mathcal{O}_X)$. For a $K3$ surface, since ω_X is trivial, $L.\omega_X = 0$. Also, $H^1(X, \mathcal{O}_X) = \mathbf{C}$ as we assume X is connected, and so by Proposition 1.4.4, $\chi(\mathcal{O}_X) = 1 - 0 + 1 = 2$, which gives the stated result.

2 Applications

Now that we have assembled powerful tools, let us apply them to prove some interesting results.

2.1 Hodge theory

William Hodge lent his name to a number of objects, among which is the Hodge decomposition. This in given in terms of Hodge numbers, which are arranged to give the Hodge diamond.

Hodge numbers

Theorem 2.1.1. [Hodge decomposition]
Let X be a $K3$ surface. Then the cohomology groups of X split as

$$H^k(X, \mathbf{C}) = \bigoplus_{p+q=k} H^{p,q}(X) \qquad \text{for} \qquad H^{p,q}(X) = H^q(X, \Omega^p(X)).$$

Th number $h^{p,q}(X) = \dim(H^{p,q}(X))$ is called the (p,q)th *Hodge number*.

This is Corollary 13.4 in Chapter I of [Bar+04], which only assumes X is a compact Kähler manifold. Also, since X has complex dimension 2, its (singular) cohomology groups exist for $k \leqslant 4$. We now apply this decomposition to calculate the Hodge numbers of $K3$ surfaces.

Lemma 2.1.2. The topological Euler characteristic of a $K3$ surface X is $\chi(X) = 24$.

Proof: A generalization of Theorem 1.4.9, called the Hirzebruch–Riemann–Roch theorem, when applied to $\mathcal{O}_X$, gives $12\chi(\mathcal{O}_X) = \omega_X.\omega_X + \chi(X)$ For this we must assume X is Kähler, which was Assertion 1.1.5. Since ω_X is trivial, $\omega_X.\omega_X = 0$. Finally, since $\mathcal{O}_X = \omega_X = \Omega^2(X)$, apply the observations of the proof of Theorem 2.1.3 to get that $\chi(\mathcal{O}_X) = 1 - 0 + 1 = 2$, and so $\chi(X) = 24$. ■

Proof: (Alternate) By Assertion 1.1.4, all $K3$ surfaces are diffeomorphic, so we will compute the topological Euler characteristic of a specific $K3$ surface. Consider the zero locus of $u^2 = x^6 + y^6 + z^6$, which we assert (without proof) is a $K3$ surface, call it Y. We may view this surface as a double cover of $\mathbf{P}^2$ (in coordinates $[x : y : z]$) ramified along the zero locus of $x^6 + y^6 + z^6$, a curve we call C. By the Riemann–Hurwitz theorem, the genus of C is $g = (d-2)(d-1)/2 = (6-2)(6-1)/2 = 10$, as C is a sextic curve. Since the topological Euler characteristic is invariant under diffeomorphism,

$$\chi(X) = \chi(Y) = 2\chi(\mathbf{P}^2) - \chi(C) = 2 \cdot 3 - (2 - 2 \cdot 10) = 6 + 18 = 24.$$

We have that $\chi(\mathbf{P}^2) = 3$ because it has exactly one cell in each dimension, and the topological Euler characteristic of a complex curve is $2 - 2g$, for g its genus. ■

Theorem 2.1.3. Let X be a $K3$ surface. The Hodge numbers of X are

$$\begin{matrix} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{matrix} \quad = \quad \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

Proof: We proceed in steps:
· $h^{2,1} = h^{1,2} = 0$ since $H^1(X, \mathbf{C}) = 0$ (as in the proof of Proposition 1.1.2) and Poincaré duality
· $h^{1,0} = h^{0,1} = 0$ by Serre duality on $(n, k) = (2, 0), (2, 1)$ and the previous step
· $h^{0,0} = h^{2,0} = 1$ since X is connected, meaning $H^0(X, \mathcal{O}_X) = \mathbf{C}$, and $\Omega^2(X) = \Omega^0(X) = \mathcal{O}_X$
· $h^{0,2} = h^{2,2} = 1$ by Serre duality on $(n, k) = (2, 2)$ and the previous step
For the last step, finding $h^{1,1}$, recall the topological Euler characteristic of a manifold is the alternating sum of the dimensions of the (co)homology groups, so in this case, $\chi(X) = 1 - 0 + 2 + h^{1,1} - 0 + 1 = 4 + h^{1,1}$. By Lemma 2.1.2, $\chi(X) = 24 = 4 + h^{1,1}$, so $h^{1,1} = 20$. ■

2.2 Current research directions

The papers [HT15] and [KKP14] use the idea of a manifold being "fibered by a $K3$ surface." A *fiber bundle* is a generalization of a vector bundle, where $\pi^{-1}(U) \cong U \times F$, for F some topological space (not necessarily a vector space or an abelian group), for $U \subset M$. More generally, a *fibration* does not require that all the fibers are the same, only that they are homotopy equivalent. Since diffeomorphism is a weaker condition, a fibration by $K3$ surfaces makes sense, in light of Assertion 1.1.4 that all $K3$ surfaces are diffeomorphic.

fibration

Both of these papers also use the *moduli space* of $K3$ surfaces. A moduli space is a new topological space where each point represents some other topological space (in this case, a $K3$ surface), and understanding the topology of the moduli space and its curves or submanifolds gives insight into the nature of $K3$ surfaces.

moduli space

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