A brief survey of $K3$ surfaces Jānis Lazovskis, 30 November 2015.

1 Punchline of the talk

The Euler characteristic of a $K3$ surface is 24 and its cohomology groups split as:

$$\begin{matrix} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{matrix} \quad = \quad \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

This is the *Hodge decomposition* of a $K3$ surface.

2 Algebraic geometry language

The name comes from the names of Kummer, Kodaira, and Kähler, as well as the mountain K2.

Defn 1. A $K3$ *surface* M is a 2-dimensional complex manifold with trivial canonical bundle and $H^1(M, \Omega^2(M)) = 0$.

The main tools will be *divisors* and *line bundles*. A divisor is the zero-section of a bundle, and under a certain equivalence, there is a one-to-one correspeondence between divisors and line bundles.

Defn 2. A *sheaf* $\mathcal{F}$ on a manifold X is an assignment of an abelian group $\mathcal{F}(U)$ to every open $U \subset X$ and maps $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for all $V \subset U$, such that

1. $\rho_{UU} = \mathrm{id}$,
2. $\rho_{VW}\rho_{UV} = \rho_{UW}$ whenever $W \subset V \subset U$,
3. for families $\{U_\alpha\}_{\alpha \in A}$ of open sets and $\{s_\alpha\}_{\alpha \in A}$ of sections, where $s_\alpha \in \mathcal{F}(U_\alpha)$, if

$$\rho_{U_\alpha, U_\alpha \cap U_\beta}(s_\alpha) = \rho_{U_\beta, U_\alpha \cap U_\beta}(s_\beta)$$

for all $\alpha, \beta \in A$, then there exists a unique $s \in \mathcal{F}(U = \bigcup_{\alpha \in A} U_\alpha)$ such that $\rho_{UU_\alpha}(s) = s_\alpha$.

Now we have *sheaf cohomology*, $H^k(M, \mathcal{F})$. Since a vector bundle is a special (*locally free*) sheaf, we also have a trivial (*invertible*) sheaf, denoted

$$\mathcal{O}_M := \{\text{sheaf of holomorphic functions on } M\}.$$

Call $\omega_M = \Omega^2(M)$ the *canonical sheaf*, so $\omega_M = \mathcal{O}_M$. Recall the *topological Euler characteristic* of M is the alternating sum of the dimensions of the (co)homology groups of M. The *holomorphic Euler characteristic* of $\mathcal{F}$ is the alternating sum of the groups $H^i(M, \mathcal{F})$. Both are denoted by the Greek letter χ.

Defn 3. Given two line bundles L_1, L_2 over a manifold M, the *intersection number* of L_1 and L_2 is

$$L_1.L_2 := \chi(\mathcal{O}_M) - \chi(L_1^*) - \chi(L_2^*) + \chi(L_1^* \otimes L_2^*).$$

3 Duality theorems

First *Poincaré duality* for regular (co)homology.

Thm 1. For M a compact oriented complex n-manifold, for $0 \leqslant k \leqslant n$,

$$H^k(M, \mathbf{C}) \cong H_{n-k}(M, \mathbf{C}).$$

Next *Serre duality* for sheaf cohomology.

Thm 2. For $\mathcal{F}$ a locally free sheaf over an n-manifold M, for $0 \leqslant k \leqslant n$,

$$H^k(M, \mathcal{F}) \cong H^{n-k}(M, \mathcal{F}^* \otimes \omega_M)^*.$$

The * indicates the *dual sheaf*, $\mathcal{F}^* := \mathrm{Hom}(\mathcal{F}, \mathcal{O}_M)$. Both of these theorems are also very useful outside the current setting.

4 Big name theorems

First the *Riemann–Roch* theorem, which is very general theorem, but here we only state it for $K3$ surfaces.

Thm 3. For L a line bundle over a $K3$ surface M,

$$\chi(L) = \frac{1}{2} L.L + 2.$$

Next the *Hodge decomposition* theorem, which applies to all Kähler manifolds.

Thm 4. The cohomology groups of a complex Kähler manifold M split as

$$H^k(M, \mathbf{C}) = \bigoplus_{p+q=k} H^q(M, \Omega^p(M)).$$

Throughout the talk we assumed two important and difficult to prove facts:

1. All $K3$ surfaces are diffeomorphic
2. All $K3$ surfaces are Kähler manifolds

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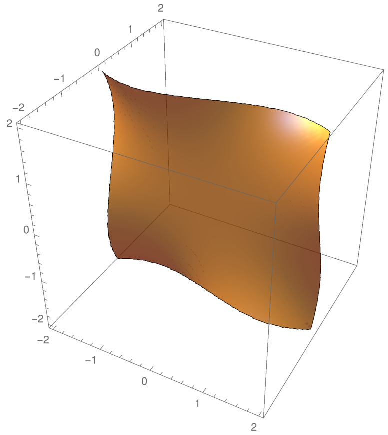
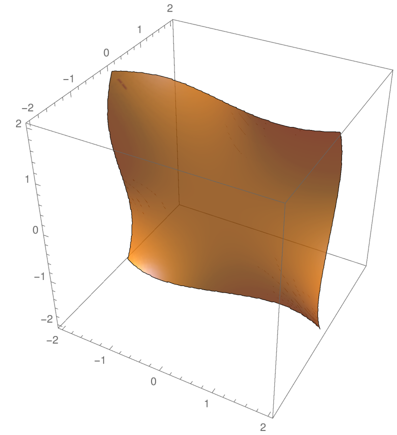




5 Visualization

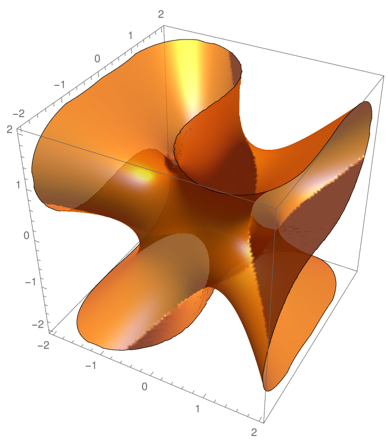
Here we have a visualization of a $K3$ surface, namely the Fermat quartic, which is the zero locus of

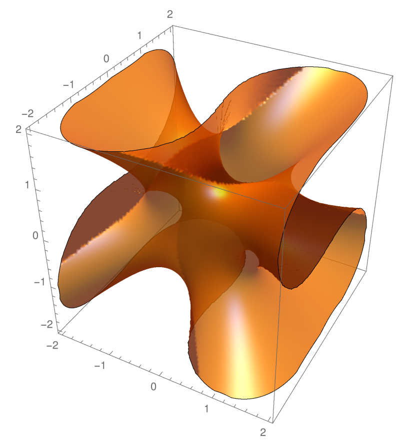
$$x^4 + y^4 + z^4 + w^4.$$

Two complex dimensions means 4 real dimensions, and so fixing the imaginary part of one of the coordinates we get the displayed images. On the left the imaginary part is less than 0, on the right it is greater than zero, and in the middle below it is zero.

An animation of a path through this surface is online at the link below.

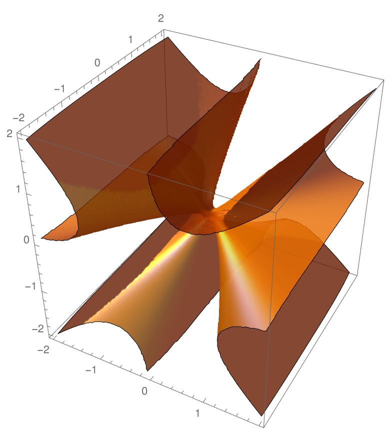
`tinyurl.com/K3surface`

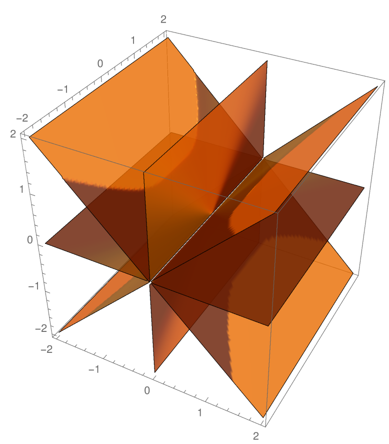


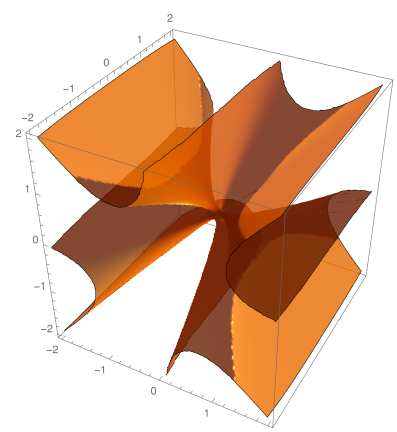


6 Problems

- Describe a $K3$ surface as a Kähler manifold.
- Show a $K3$ surface is not the blowup of any other smooth surface.
- Construct the Hodge–de Rham spectral sequence for a $K3$ surface.







7 References

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The full survey is available online at `math.uic.edu/~jlv/docs/2015-11-30-diffmani.pdf`