Math 549 exercises

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1. (August 24) Find an atlas on the extended complex plane $\mathbf{C} \cup \{\infty\}$.
2. (August 24) Find an atlas on the real projective space $\mathbf{RP}^n = \{$1-dimensional subspaces of $\mathbf{R}^n\}$.
3. (August 28) Show that the stereographic projection $\pi : S^2 \setminus \{N\} \to \mathbf{R}^2$ is a diffeomorphism, for N the "north pole" of the sphere S^2.
4. (August 28) Show that $O(n)$, the space of orthogonal $n \times n$ matrices, and $SO(n)$, the space of orthogonal matrices with determinant 1, are both manifolds.
5. (August 31) Show that a smooth map of manifolds is continuous, using the topology of the manifolds.
6. (August 31) Show that $SO(3)$ is diffeomorphic to $\mathbf{RP}^3$.
7. (September 2) Show that $C^\infty(M)$, the space of smooth maps $M \to \mathbf{R}$, is a vector space.
8. (September 2) Describe an n-dimensional analogue of the smooth bump function presented in class.
9. (September 4) Let $M \ni a$ be a n-dimensional manifold in coordinates $x_1, \ldots, x_n$. Show that $(dx_1)_a, \ldots, (dx_n)_a$ span T_a^*M.
10. (September 9) Find a basis for T_pS^3, the tangent space of S^3 at a point p.
11. (September 11) Prove the following statement: Let $F : M \to N$ be a smooth map and $c \in N$ such that for all $a \in F^{-1}(c)$, the derivative DF_a is surjective. Then $F^{-1}(c)$ is a smooth manifold of dimension $\dim(M) - \dim(N)$.
12. (September 11) Let $f : M \to N$ be a diffeomorphism of manifolds. Show that for each $x \in M$, $(Df)_x$ is an isomorphism of tangent spaces.
13. (September 11) Let X be a manifold with $U \subset X$ open. Show that $T_aU = T_aX$ for all $a \in U$.
14. (September 14) Consider the map $i : (-1, \infty) \to \mathbf{R}^2$ given by $t \mapsto (t^2 - 1, t(t^2 - 1))$. Show that this map does not give a submanifold of $\mathbf{R}^2$.
15. (September 14) Let $M \ni x, N \ni y$ be two manifolds. Show that $T_{(x,y)}M \times N \cong T_xM \times T_yN$.
16. (September 18) Show that the 1-sphere S^1 has trivial tangent bundle.
17. (September 18) Prove the following statement: A manifold M^n has trivial tangent bundle iff there are n vector fields $X_1, \ldots, X_n$ on M such that at each $a \in M$, the elements $(X_1)_a, \ldots, (X_n)_a$ form a basis for T_aM.
18. (September 18) Prove the following statement: Any linear transformation which satisfies the Leibniz property is a vector field.
19. (September 18) Let X, Y, Z be vector fields on a manifold M. Show the following properties hold, in coordinates:
    (a) $[X, Y + Z] = [X, Y] + [X, Z]$
    (b) $[X, Y] = -[Y, X]$
    (c) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$
    (d) $\lambda[X, Y] = [X, \lambda Y]$ for any scalar λ
20. (September 21) Let A be a skew-symmetric $m \times m$ matrix, and set $\gamma(t) = \exp(tA) = \sum_{n=0}^{\infty} t^n A^n / n!$.
    (a) Show that γ defines a smooth curve in $SO(m)$.
    (b) Find $\gamma'(0)$, the tangent vector defined by γ at 0.
    (c) Find $T_I SO(m)$.
    (d) Find $T_g SO(m)$, for arbitrary $g \in SO(m)$.

21. (October 12) Show that a smooth vector field on a manifold M that vanishes outside a compact set $K \subset M$ generates a 1-parameter group of diffeomorphisms on M.

22. (October 14) Consider $S^2 \subset \mathbf{R}^3$ in coordinates (x, y, z), and let $X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ and $Y = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$ be vector fields on S^2. Calculate $[X, Y]$.

23. (October 16) Let X, Y be vector fields on a smooth manifold M. Give the definition of the Lie bracket $[X, Y]$ as a differential operator on smooth functions. Also show that $L_{[X,Y]} = [L_X, L_Y]$, for $L_X Y = [X, Y]$ the *Lie derivative*.

24. (October 16) Let $v_1, \dots, v_n$ be a basis of an n-dimensional vector space V. Show that the elements $v_{i_1} \wedge \cdots \wedge v_{i_p}$, for $1 \leqslant i_1 < \cdots < i_p \leqslant n$, form a basis for $\bigwedge^p V$.

25. (October 16) For $n \geqslant 1$, show that $SL(n)$ is a smooth manifold, and find its dimension.

26. (October 16) Let M, N be smooth manifolds with $M \subset N$ a submanifold. Show that if X is a vector field defined on an open neighborhood of M, then there exists a vector field Y on N such that $Y|_M = X|_M$.

27. (October 21) Show that every compact manifold has a vector field with finitely many zeros.

28. (October 21) Calculate $F^*\alpha$ for $F : \mathbf{R}^3 \to \mathbf{R}^2$ given by $F(x_1, x_2, x_3) = (x_1x_2, x_2 + x_3)$ and $\alpha = x dx \wedge dy$.

29. (October 23) Let M be a smooth n-manifold and ω a k-form on M. Give $d\omega$ in local coordinates and show why it is independent of the basis chosen for M.

30. (October 26) Let $U \subset \mathbf{R}^n$, $V \subset \mathbf{R}^m$ be open sets with coordinates x_i, y_i, respectively, and $\theta : U \to V$ be a smooth map. Show that, for $\theta_i = y_i \circ \theta$,
$$\theta^*(dy_i) = \frac{\partial \theta_i}{\partial x_j} dx_j.$$

31. (October 26) Define the *Hodge star* operator
$$\begin{array}{rcl} * \;:\; \Omega^k(\mathbf{R}^m) & \to & \Omega^{m-k}(\mathbf{R}^m), \\ dx_{i_1} \wedge \cdots \wedge dx_{i_k} & \mapsto & \operatorname{sgn}(\sigma) dx_{j_1} \wedge \cdots \wedge dx_{j_{m-k}}, \end{array}$$
with $1 \leqslant i_1 < \cdots < i_k \leqslant m$ and $1 \leqslant j_1 < \cdots < j_{m-k} \leqslant m$. Also $\{i_1, \dots, i_k, j_1, \dots, j_{m-k}\} = \{1, \dots, m\}$ and σ is the permutation $(i_1 \; \cdots \; i_k \; j_1 \; \cdots \; j_{m-k}) \in S_m$ (the symmetric group on m elements). Let $\omega = a_{12} dx_1 \wedge dx_2 + a_{13} dx_1 \wedge dx_3 + a_{23} dx_2 \wedge dx_3$.
    (a) Calculate $*\omega$ for $\omega \in \Omega^2(\mathbf{R}^3)$.
    (b) Calculate $*\omega$ for $\omega \in \Omega^2(\mathbf{R}^4)$.

32. (October 26) Show that the formula $\mathscr{L}_X \alpha = d(i_X \alpha) + i_X(d\alpha)$ agrees with the definition of $\mathscr{L}_X \alpha$.

33. (October 28) Let $F : M \times [0, 1] \to N$ be a smooth map and $\alpha \in H^p(N)$. Give a description of $F^*\alpha = \beta + dt \wedge \gamma$ in local coordinates.

34. (October 30) Let M be a smooth manifold. Show that $H^p(M \times \mathbf{R}^n) \cong H^p(M)$ for any p. This result is known as *Poincare's lemma*.

35. (October 30) Prove that $H^p(S^n) = \mathbf{R}$ if $p = 0, n$ and 0 otherwise. You may assume the result for $n = 1$.

36. (November 2) Consider the space of straight lines in $\mathbf{R}^3$.
    (a) Describe this space as a manifold.
    (b) What is the dimension of this manifold?
    (c) Show this manifold is not orientable.

37. (November 2) Prove that the tangent bundle of a smooth manifold is orientable.

38. (November 2) Let α be a smooth 1-form on $\mathbf{R}^2$. Show that α is exact if and only if it is closed.

39. (November 2) Let M be the complement of the origin in $\mathbf{R}^3$. Construct a 2-form on M which is closed but not exact.

40. (November 4) Construct a smooth map $f : S^2 \to \mathbf{RP}^2$ and show, by contradiction, that $\mathbf{RP}^2$ is not orientable (by pulling back an orientation form on $\mathbf{RP}^2$ to an orientation form on S^2).

41. (November 6) Let M, N be smooth manifolds of dimension n, and $f : M \to N$ a smooth bijective immersion. Show that f is a diffeomorphism.

42. (November 6) Let M be a connected manifold without boundary. Show that if S, T are finite sets in M of the same size, then there is a diffeomorphism $f : M \to M$ sending S to T (that is, $f(S) = T$).

43. (November 6) Let M be a compact smooth orientable n-manifold. Show that there exists a smooth map $f : M \to S^n$ of non-zero degree.

44. (November 9) Let $P \subset \mathbf{R}^3$ be a finite set. Show that there is a smooth embedding $f : S^2 \to \mathbf{R}^3$ such that $P \subset f(S^2)$.

45. (November 9) Let C be a closed curve in $\mathbf{R}^2$, given by the zero locus of $f(x, y)$, and $\omega = xdy$ a 1-form on $\mathbf{R}^2$. Show that the integral of ω over f is equal to the area enclosed by the curve.

46. (November 9) Describe an equivalent statement to the exercise above, but for surfaces in $\mathbf{R}^3$.

47. (November 9) Let $M \ni x$ be an n-manifold without boundary and $B(x) \subseteq M$ a closed neighborhood of x diffeomorphic to the unit n-ball. Prove that $M - \{x\}$ is diffeomorphic to $M - B(x)$.

48. (November 11) Show that $S^1 \times S^1$ is not diffeomorphic to S^2.

49. (November 13) Let M be a manifold with boundary ∂M. Show that an orientation M defines an orientation on ∂M.

50. (November 13) Let M be a compact orientable manifold with boundary ∂M. Recall that a *retract* of M onto a subset $N \subset M$ is a continuous map $r : M \to N$ such that $r(n) = n$ for all $n \in N$. Show that there is no smooth retract $M \to \partial M$.