1 Negative cyclic homology

k is a commutative ring
A is an associative k-algebra
d_i is the ith degeneracy map $(a_0, \ldots, a_n) \mapsto (a_0, \ldots, a_i a_{i+1}, \ldots, a_n)$
b_n, b'_n are Hochschild boundary maps:

b'_n is b_n forgetting d_n

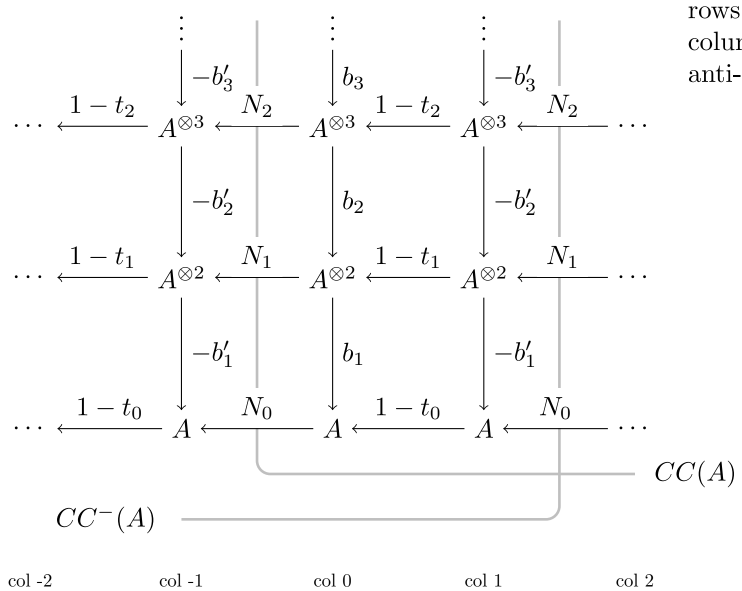
$$b_n = \sum_{i=0}^{n} (-1)^i d_i \;:\; A^{\otimes n+1} \to A^{\otimes n}$$

Define two new maps, using the *cyclic operator* t_n:

subscript n is omitted when clear

$$\begin{aligned} t_n \;:\; A^{\otimes n+1} &\to A^{n+1}, \\ (a_0, \ldots, a_n) &\mapsto (a_n, a_0, \ldots, a_{n-1}), \end{aligned} \qquad \begin{aligned} N_n \;:\; A^{\otimes n+1} &\to A^{n+1}, \\ (a_0, \ldots, a_n) &\mapsto (1 + t_n + t_n^2 + \cdots + t_n^n)(a_0, \ldots, a_n). \end{aligned}$$

Prop: There exists an anti-commuting upper half-plane double complex as below.



rows: $t_n^{n+1} = 1$
columns: from Hochschild cpx
anti-comm: algebra

Def: The nth *cyclic homology* group of A is

$$HC_n(A) := H_n(TCC(A)).$$

$$T_k CC(A) = \bigoplus_{p+q=k} CC_{pq}(A)$$

The nth *negative cyclic homology* group of A is

$$HC_n^-(A) := H_n(T^{\prod} CC^-(A)).$$

$$T_k^{\prod} CC^-(A)) = \prod_{p+q=k} CC_{pq}(A)$$

Have to take product for HC^- because have infinitely many factors in each total object

Example: Let $A = k$, $k^{\otimes \ell} = k \otimes_k \cdots \otimes_k k = k$. Then

$$\begin{aligned} 1 - t_n &= 0 \\ N_n &= n+1 \end{aligned} \qquad b_n = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases} \qquad -b'_n = \begin{cases} -1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Hence

non-zero homology comes from second row

$$HC_n^-(k) = \begin{cases} 0 & n > 0 \text{ or odd,} \\ k & n \text{ even.} \end{cases}$$

A is smooth as a k-algebra if Spec(A) is smooth as a variety

Example: Let $k = \mathbf{Q}$, $A = \mathbf{Q}[t]$. $\mathbf{Q}[t]$ is a smooth commutative $\mathbf{Q}$-algebra.

- **Thm:** (Hochschild, Kostant, Rosenberg 1962) For A a smooth k-algebra, $HH_*(A) \cong \Omega^*_{A/k}$.
- (Connes) $\exists$ LES relating HH and HC
- $\exists$ spectral sequence of de Rham cohomology groups differential forms abutting to HC

$$HC_n^-(\mathbf{Q}[t]) = \ker(\delta_n) \times \prod_{i>0} H_{dR}^{n+2i}(\mathbf{Q}[t]).$$

$\delta_n : \Omega^n_{\mathbf{Q}[t]/\mathbf{Q}} \to \Omega^{n+1}_{\mathbf{Q}[t]/\mathbf{Q}}$ is the exterior derivative

$H^k_{dR}(\mathbf{Q}[t])$ is the de Rham cohomology

$\Omega^1_{A/k}$ is ring of Kähler differentials (formal derivations)

2 K-theory

Def: *K-theory* is a functor Ring$\to$Spec that takes a ring R to its spectrum $K(R)$.
$K_n(R) = \pi_n K(R)$.

set of prime ideals

Example: Let $R = F_p$. Then

rationalize

$$K_n(F_p) = \begin{cases} \mathbf{Z} & n = 0, \\ \mathbf{Z}/(p^{i+1}-1)\mathbf{Z} & n = 2i+1, \\ 0 & n = 2i, \end{cases} \qquad \text{so} \qquad K_n(F_p) \otimes_{\mathbf{Z}} \mathbf{Q} = \begin{cases} \mathbf{Q} & n = 0, \\ 0 & \text{else.} \end{cases}$$

3 Goodwillie's theorem

Def: Let $f : R \to S$ be a homomorphism of (simplicial) rings. The groups $K_n(f)$ and $HC_n^-(f)$ are defined to be the groups such that the sequences

$$\cdots \longrightarrow K_n(R) \longrightarrow K_n(S) \longrightarrow K_n(f) \longrightarrow K_{n-1}(R) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow HC_n^-(R) \longrightarrow HC_n^-(S) \longrightarrow HC_n^-(f) \longrightarrow HC_{n-1}^-(R) \longrightarrow \cdots$$

are exact.

since K, HC are functors, get induced maps
knowing all but one group is enough to figure out group

Thm:(Goodwillie 1986) Let $f : R \to S$ be a surjective homomorphism of (simplicial) rings with nilpotent kernel. Then

$$K_n(f) \otimes_{\mathbf{Z}} \mathbf{Q} \cong HC_{n-1}^-(f) \otimes_{\mathbf{Z}} \mathbf{Q}$$

for all $n \in \mathbf{Z}_{\geqslant 0}$.

Example: For $k = \mathbf{Q}$ and $A = \mathbf{Q}[t]/(t^2)$, we know

$$HC_n^-(\mathbf{Q}[t]/(t^2)) = \begin{cases} 0 & n \text{ odd}, \\ \mathbf{Q} & n \text{ even}. \end{cases}$$

The map $f : \mathbf{Q}[t]/(t^2) \to \mathbf{Q}$ given by $t \to 0$ is surjective, so we know $HC_n^-(f)$. The kernel $t\mathbf{Q}$ is nilpotent, so by Goodwillie, we know $K_n(f)$, and so we know $K_n(\mathbf{Q}[t]/(t^2))$.