Sheaf cohomology may be used to calculate invariants of groups and dimensions (relevant to geometric group theory).

1 Setting

Def: Let X be a topological space. A *presheaf* $\mathcal{F}$ on X is:

1. an assignment of an abelian group $\mathcal{F}(U)$ to every open $U \subset X$

2. maps $\rho_{UV} : U \to V$ for all $V \subset U$

such that:

1. $\mathcal{F}(\emptyset) = 0$,

2. $\rho_{UU} = \text{id}$,

3. $\rho_{VW}\rho_{UV} = \rho_{UW}$ whenever $W \subset V \subset U$.

- can be in any cat, not just Top
- can be in any cat, not just Ab
- elements of $\mathcal{F}(U)$ are *sections*
- elements ρ_{UV} are *restriction maps*

Eg: Some examples of presheaves:

1. $\mathcal{O}(U) = \{\text{differentiable functions } U \to \mathbb{R}\}$. ρ is function restriction

2. $\underline{A}(U) = \{\text{(locally) constant } A\text{-valued functions } U \to \mathbb{R}\}$. ρ is identity on connected components

3. skyscraper sheaf $\mathcal{S}(U) = \begin{cases} A & \text{if } x \in U, \\ \emptyset & \text{if } x \notin U. \end{cases}$. ρ is identity or 0.

Def: A *sheaf* $\mathcal{F}$ on X is a presheaf on X such that for any family $\{U_\alpha\}_{\alpha \in A}$ of open sets $U_\alpha \subset X$ and any family $\{s_\alpha\}_{\alpha \in A}$ of sections $s_\alpha \in \mathcal{F}(U_\alpha)$, if

$$\rho_{U_\alpha, U_\alpha \cap U_\beta}(s_\alpha) = \rho_{U_\beta, U_\alpha \cap U_\beta}(s_\beta)$$

for all $\alpha, \beta \in A$, then there exists a unique $s \in \mathcal{F}(U = \bigcup_{\alpha \in A} U_\alpha)$ such that $\rho_{UU_\alpha}(s) = s_\alpha$. *gluing axiom*

Rem: $\mathcal{O}(X)$ and $\mathcal{S}(X)$ are sheaves, but the constant sheaf $\underline{A}(U)$ is not. Consider $A = \mathbb{Z}$, $X = \mathbb{R}$, $U_0 = (0,1)$ and $U_1 = (2,3)$, with $s_0 = 0 \in \underline{\mathbb{Z}}(U_0)$ and $s_1 = 1 \in \underline{\mathbb{Z}}(U_1)$. Since $U_0 \cap U_1 = \emptyset$, the restriction map condition is satisfied. However, there exists no $s \in \underline{\mathbb{Z}}(U)$ that restricts to both 0 and 1.

Def: A *stalk* of a sheaf $\mathcal{F}$ at $x \in X$ is the direct limit

$$\mathcal{F}_x := \lim_{x \in U} \mathcal{F}(U) = \bigoplus_{x \in U} \mathcal{F}(U) \Big/ \begin{matrix} \iota_V \rho_{UV}(s) \sim \iota_U(s) \\ \forall\ s \in \mathcal{F}(U), V \subset U, \end{matrix}$$

where $i_U : U \to \bigoplus_{x \in U} \mathcal{F}(U)$ is the natural inclusion map. A *germ* of a stalk is the equivalence class of a section under the quotient above.

Def: Given a presheaf $\mathcal{F}$, there is a sheaf $\mathcal{F}^+$ associated to it, called the *sheafification* of $\mathcal{F}$. Set

$$\mathcal{F}^+(U) := \left\{ s \in \bigoplus_{t \in U} \mathcal{F}_t \ : \ \forall\ q \in U,\ \exists \begin{matrix} \cdot \text{ a nbhd } V \subset U \text{ of } q \\ \cdot \text{ a section } \widetilde{s} \in \mathcal{F}(V) \end{matrix} \ \text{such that } s\big|_{\mathcal{F}_v} = \widetilde{s}\big|_{\mathcal{F}_v}\ \forall\ v \in V \right\}$$

for every open set U. The restriction maps are induced naturally from $\bigoplus_{t \in U} \mathcal{F}(U) \to \bigoplus_{t \in V} \mathcal{F}(V)$.

Now we may always talk about sheaves only satisfying the three conditions given first above.

2 Cohomology

Given sheaves $\mathcal{F}$, $\mathcal{G}$ both on X, we may define a morphism $f : \mathcal{F} \to \mathcal{G}$ by maps $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\rho^{\mathcal{F}}_{UV}} & \mathcal{F}(V) \\ {\scriptstyle f_U}\downarrow & & \downarrow{\scriptstyle f_V} \\ \mathcal{G}(U) & \xrightarrow{\rho^{\mathcal{G}}_{UV}} & \mathcal{G}(V) \end{array}$$

commutes for every $V \subset U$. This gives structure of category on sheaves

Given a sheaf $\mathcal{F}$ on X, we may calculate cohomology groups $H^i(X, \mathcal{F})$ of X with values in $\mathcal{F}$ by
· cocycles modulo coboundaries
· derived functors of global section functor $\Gamma(X, -) : Sh(X) \to Ab$
Need exact sequences of sheaves.

Thm: A sequence of sheaves is exact iff it is exact on the stalks.

Calculating cohomology is difficult. Try to create *flabby* (flasque) sheaves (ρ_{XU} is surjective for all $U \subset X$) for which $H^i = 0$ for all $i > 0$.

Eg: Consider the following exact sequence of sheaves on a complex manifold X:

$$0 \longrightarrow \underline{\mathbb{Z}}(X) \xrightarrow{i} \mathcal{O}(X) \xrightarrow{\exp} \mathcal{O}^*(X) \longrightarrow 0.$$

sheaf of locally constant integer-valued functions ($\underline{\mathbb{Z}}(X)$)
sheaf of complex diffb'le functions ($\mathcal{O}(X)$)
sheaf of non-zero complex diffb'le functions ($\mathcal{O}^*(X)$)

Check it is exact by checking exactness on the stalks. Let $X = S^2 = \mathbb{C}P^1$ and take cohomology. Interesting fact - connecting map δ below takes line bundle ($H^1(X; \mathcal{O}^*(X))$ classifies line bundles on X) to its first Chern class:

$$\cdots \longrightarrow H^1(X; \mathcal{O}(X)) \longrightarrow H^1(X; \mathcal{O}^*(X) \xrightarrow{\delta} H^2(X; \underline{\mathbb{Z}}(X)) \longrightarrow \cdots$$