Curvature of principal bundles and their Euler classes

M is smooth, compact $2n$-dimensional manifold

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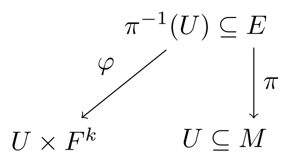
vector bundle: M a topological manifold, F a field. A *F-vector bundle over M* of rank k is a triple (E, M, π), usually denoted by just E, where

1. E is a topological space called the *total space*,
2. $\pi : E \to M$ is a continuous map called the *projection map*,
3. $\forall\ p \in M$, the *fiber* $E_p := \pi^{-1}(p)$ has a vector space structure, and
4. $\forall\ p \in M$, $\exists$ a neighborhood $U \ni p$, a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F^k$ so that $v \in \pi^{-1}(\{p\}) \mapsto \varphi(p, v) \in F^k$ is a linear isomorphism (*local triviality* condition).



0.0.1 Intro

> *Examples of vector bundles:* **DRAW PICTURES**
>
> tangent bundle on S^1 Mobius bundle on S^1

Goal:
· Give explicit constructions of cohomology classes that determine isomorphism class of 2n-dim cpt mflds
  · up to finite number of possibilites, that admit principal $GL^+(2n)$-bundle

Assumption:
· Euler number $e(M)$ is invariant of bundles

0.0.2 Euler number

· A *characteristic class* of a vector bundle E is a cohomology class $c \in H^*(M; R)$.
  · R is some ring
  · Related to E because class is pullback from $H^*(E; R)$ induced from projection map π
  · Euler (denoted e), Chern, Pontryagin, Thom, Stiefel–Whitney, etc.

· The *fundamental class* $[M]$ of M compact, connected, orientable, is a generator for $H_n(M; \mathbb{Z}) = \mathbb{Z}$.
  · $+1$ and -1 are only choices

· The *Euler number* $e(M)$ of a vector bundle E is the evaluation of e on $[M]$, in this case $\int_M e$
  · Works because Euler class is element in poly ring. Maybe inner prod
  · When $E = TM$, $e(M) = \chi(M)$ (the Euler number is the Euler characteristic of M)

· Euler number of odd-dim mflds is zero
  · Follows from universal coefficient theorem

0.0.3 Principal bundles

· A *principal bundle* is a vector bundle with a group G (*structure group*) and a G-action $\rho : E \times G \to E$
  · such that $\pi : E \to M$ is isomorphic to the quotient map $E \to E/G$. Also fiber bundle
  · $GL(k)$, $O(k)$, $U(k)$, Lie groups

> *Example of principal bundle:* **DRAW PICTURE** - sphere with tangent bundle
> Given M^n, $E_p = \{\text{ordered bases of } T_pM\} = \{\text{isomorphisms } \mathbb{R}^n \to T_pM\}$, $G = GL(n, \mathbb{R})$
> Called a 'frame bundle'

· A *connection* on a principal G-bundle is a special $\mathfrak{g}$-valued 1-form ω on E
  · $\mathfrak{g}$ is the Lie algebra of G
  · The *curvature* of ω is a special $\mathfrak{g}$-valued 2-form on E ($d\omega + \frac{1}{2}(\omega \wedge \omega)$)
  · A *flat connection* has curvature zero
  · We are interested in manifolds with flat connections

0.0.4 Milnor and Sullivan

Thm: (Milnor, 1957)
For $g > 0$, $|e(M^2)| \geqslant g$ iff the $GL^+(2)$-bundle over M^2 does not have a connection with curvature zero.
  · Will talk about forward direction

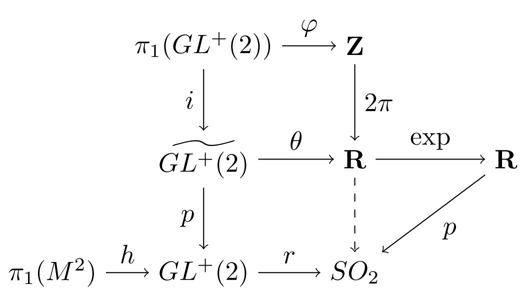
Thm: (Sullivan, 1975)
If the $GL^+(2n)$-bundle over M^{2n} has a connection with curvature zero, then $|e(M^{2n})| < k_M$.
  · Hence the name 'bounded' cohomology

· For every M, G, there is a map h and an isomorphism

$$[h : \pi_1(M) \to G] \cong H^{\dim(M)}(M, \pi_1(G))$$

  · image of h is the *holonomy group* **DRAW PICTURE** - sphere with $90 - 90 - 90$ triangle, rot
    · elements linear transformations of T_pM, operation multiplication (composition of loops)
  · isomorphism from Hurewicz theorem

· Proof uses commutative diagram: pieraksti lietas pa labi uz tafeles cita krasa



· r is take rotational component of map
  · matrix in $GL^+(2)$ is rotation and scaling
· $\widetilde{GL^+}(2)$ is universal cover of G
  · is $\mathbf{R}^3$ because $SL(2) \cong S^1 \times \mathbf{R}^2$
· θ is take angle with multiplicity
· column exact because $\ker(p) = \pi_1(GL^+(2))$
· φ sends to generator to Euler class (± 1)
· multiply by 2π to make commute
· $\mathbf{R}$ is universal cover of SO_2, since SO_2 is circle
· apply exp to make commute
  · why dashed vert arr?

· φ carries obstruction class from $H^2(M^2; \pi_1(GL^+(2)))$ into Euler class.